



Algebraic geometry

The class of the affine line is a zero divisor in the Grothendieck ring: An improvement

*La classe de la droite affine est un diviseur de zéro dans l'anneau de Grothendieck : une amélioration*

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ABSTRACT

Lev A. Borisov has shown that the class of the affine line is a zero divisor in the Grothendieck ring of algebraic varieties over complex numbers. We improve the final formula by removing a factor.

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R É S U M É

Lev A. Borisov a prouvé que la classe de la droite affine est un diviseur de zéro dans l'anneau de Grothendieck des variétés algébriques complexes. Nous améliorons la formule finale en supprimant un facteur.

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1. Introduction

The Grothendieck ring $K_0(\text{Var}_{\mathbb{C}})$ of complex algebraic varieties is defined as the quotient of the free abelian group generated by the isomorphism classes $[X]$ of complex algebraic varieties modulo the relations

$$[X] = [Y] + [X \setminus Y]$$

for all closed subvarieties $Y \subset X$. The Cartesian product of varieties gives the product structure.

The class $\mathbb{L} = [\mathbb{A}^1(\mathbb{C})]$ of the affine line has a major role in the study of the Grothendieck ring. It has been proved in [2] that X and Y are stably birational if and only if their classes $[X]$ and $[Y]$ are equal modulo $\mathbb{L}$. After Bjorn Poonen had shown in [3] that $K_0(\text{Var}_{\mathbb{C}})$ is not a domain, Lev Borisov has clarified this result in [1] by showing that $\mathbb{L}$ is a zero divisor. He has compared the two sides $[X_W]$ and $[Y_W]$ of the Pfaffian–Grassmannian double mirror correspondence, and obtained the following formula:

$$([X_W] - [Y_W]) \cdot (\mathbb{L}^2 - 1) \cdot (\mathbb{L} - 1) \cdot \mathbb{L}^7 = 0.$$

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This result is not only an improvement of that of Poonen: it is crucial in motivic integration to understand the kernel of the localization morphism $K_0(\text{Var}_{\mathbb{C}}) \rightarrow K_0(\text{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$, since we consider classes in the localized ring. In this paper, we improve this formula as follows.

Theorem 1.1. $([X_W] - [Y_W]) \cdot \mathbb{L}^6 = 0$.

2. The class of Grassmannians

Proposition 2.1. For $2 \leq k < n$, we have the relation

$$[G(k, n)] = [G(k, n-1)] + \mathbb{L}^{n-k} \cdot [G(k-1, n-1)].$$

Proof. Let $e_1, \dots, e_n$ be the canonical basis of $\mathbb{C}^n$, F the hyperplane orthogonal to e_n , $U \subset G(k, n)$ the open subset defined by $\{T \in G(k, n) \mid \dim(T \cap F) = k-1\}$ and $\pi: U \rightarrow G(k-1, F)$ the regular mapping that sends T on $T \cap F$. For $S \in G(k-1, F)$, the fiber $\pi^{-1}(S)$ can be identified with

$$\mathbb{P}(\mathbb{C}^n/S) \setminus \mathbb{P}(F/S) \simeq \mathbb{A}^{n-k}.$$

Let H be a complementary subspace of S in F and the open subset $V = \{S' \in G(k-1, F) \mid S' \oplus H = F\}$. For all $S' \in V$, we have the identification $\mathbb{C}^n/S' \simeq H \oplus \mathbb{C}e_n$, hence π is a trivial fibration over V . Consequently, π is a locally trivial fibration, therefore $[U] = \mathbb{L}^{n-k} \cdot [G(k-1, n-1)]$. We have $[G(k, n)] = [Z] + [U]$ with $Z = G(k, n) \setminus U = \{T \in G(k, n) \mid T \subset F\} = G(k, F)$, which shows the announced formula. $\square$

A simple induction gives the following formulas for $n \geq 4$:

$$[G(2, n)] = \begin{cases} [\mathbb{P}^{n-2}] \cdot \sum_{k=0}^{(n-2)/2} \mathbb{L}^{2k} & \text{if } n \text{ is even} \\ [\mathbb{P}^{n-1}] \cdot \sum_{k=0}^{(n-3)/2} \mathbb{L}^{2k} & \text{if } n \text{ is odd.} \end{cases}$$

For example, $[G(2, 5)] = [\mathbb{P}^4] \cdot (\mathbb{L}^2 + 1)$ and $[G(2, 7)] = [\mathbb{P}^6] \cdot (\mathbb{L}^4 + \mathbb{L}^2 + 1)$.

3. Improvement of Borisov's formula

3.1. Pfaffian and Grassmannian double mirror varieties

Let V be a 7-dimensional complex vector space and W a generic 7-dimensional space of skew forms on V . We define X_W as a subvariety of the Grassmannian $G(2, V)$, which is the locus of all $T \in G(2, V)$ with $\omega|_T = 0$ for all $\omega \in W$, and Y_W as a subvariety of $\mathbb{P}W$ of skew forms whose rank is less than 6. Smoothness of these two varieties has been shown by E. Rødland in [4]. Furthermore, we know that all forms in Y_W have rank 4 and all forms in $\mathbb{P}W \setminus Y_W$ have rank 6.

3.2. The formula

Let us define H as a subvariety of $G(2, V) \times \mathbb{P}W$ that consists of pairs $(T, \mathbb{C}\omega)$ with $\omega|_T = 0$. In order to obtain the explicit equations that define H , let us set $T_0 \in G(2, V)$ with basis e_1, e_2 and H a complementary subspace with basis $e_3, \dots, e_7$. The neighborhood $U = \{T \in G(2, V) \mid T \oplus H = V\}$ of T_0 can be identified with $\mathcal{L}(T_0, H)$ by considering the map $f \in \mathcal{L}(T_0, H) \mapsto \{x + f(x) \mid x \in T_0\} \in U$. If we set $(f_{i,j})_{(i,j) \in \{1,2\} \times \{3,\dots,7\}}$ the basis of $\mathcal{L}(T_0, H)$ adapted to the two bases previously considered, we can identify $T \in U$ with $\{x + \sum \alpha_{i,j} f_{i,j}(x) \mid x \in T_0\}$. Now, for $\omega = \sum_{i=1}^7 \beta_i \omega_i \in W$, the condition $\omega|_T = 0$ can be expressed as

$$\sum_{i=1}^7 \beta_i \omega_i \left(e_1 + \sum_{j=3}^7 \alpha_{1,j} e_j, e_2 + \sum_{j=3}^7 \alpha_{2,j} e_j \right) = 0.$$

Looking at the projections onto the two factors $G(2, V)$ and $\mathbb{P}W$ will give us two ways to express $[H]$. Theorem 1.1 will be a direct consequence of the two next propositions.

Proposition 3.1. $[H] = [\mathbb{P}^6] \cdot (\mathbb{L}^4 + \mathbb{L}^2 + 1) \cdot [\mathbb{P}^5] + [X_W] \cdot \mathbb{L}^6$.

Proof. Considering the projection $p : H \rightarrow G(2, V)$ onto the first factor, which is a trivial fibration in restriction to $p^{-1}(X_W)$ and a locally trivial fibration in restriction to $G(2, V) \setminus p^{-1}(X_W)$, Proposition 2.4 of [1] proves that

$$[H] = [G(2, 7)] \cdot [\mathbb{P}^5] + [X_W] \cdot \mathbb{L}^6.$$

The expression $[G(2, 7)] = [\mathbb{P}^6] \cdot (\mathbb{L}^4 + \mathbb{L}^2 + 1)$ gives the result. $\square$

Proposition 3.2. $[H] = [Y_W] \cdot \mathbb{L}^6 + [\mathbb{P}^6] \cdot [\mathbb{P}^5] \cdot (\mathbb{L}^4 + \mathbb{L}^2 + 1)$.

Lemma 3.3. Let $\pi : H \rightarrow \mathbb{P}W$ be the projection onto the second factor. Its restrictions to $\pi^{-1}(Y_W)$ and $\pi^{-1}(\mathbb{P}W \setminus Y_W)$ are piecewise trivial fibrations (see 4.2.1 in [5]).

Proof of the lemma. The reasoning is the same for rank 4 ($Y_4 = Y_W$) and rank 6 ($Y_6 = \mathbb{P}W \setminus Y_W$). For $i \in \{4, 6\}$, let us set

$$Z_i = \pi^{-1}(Y_i) = H \cap (G(2, V) \times Y_i).$$

In order to have piecewise triviality of π on Z_i , it suffices, according to Theorem 4.2.3 in [5], to prove that there exists a uniform fiber F_i such that for all $x \in Y_i$,

$$Z_i \times_{Y_i} \{x\} \simeq F_i \times_{\mathbb{C}} \text{Spec}(\kappa(x)).$$

To achieve this, it suffices to note that a skew form of rank 4 or 6 with coefficients in a field $K \supset \mathbb{C}$ is congruent with the skew form

$$\begin{pmatrix} 0 & I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & I_3 & 0 \\ -I_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with a base change having coefficients in K , an action that spreads on fibers. $\square$

Lemma 3.4. Let $\mathbb{C}\omega \in Y_W$ be a closed point. Then the class of its fiber is

$$[\pi^{-1}(\mathbb{C}\omega)] = [\mathbb{P}^5] \cdot (\mathbb{L}^4 + \mathbb{L}^2 + 1) + \mathbb{L}^6.$$

Proof. As $\text{rk}(\omega) = 4$, there exists a basis $e_1, \dots, e_7$ of V in which the matrix of ω is

$$\begin{pmatrix} 0 & I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Denote $F = \text{Vect}\{e_3, \dots, e_7\}$ and $H = F \oplus \mathbb{C}e_2$. We have

$$[\pi^{-1}(\mathbb{C}\omega)] = [\{T \in G(2, V) \mid \omega|_T = 0\}] = [\{T \in G(2, H) \mid \omega|_T = 0\}] + [U]$$

where U is the open subset $\{T \in G(2, V) \mid \dim(T \cap H) = 1, \omega|_T = 0\}$, with the locally trivial fibration $\pi : U \rightarrow \mathbb{P}H = \mathbb{P}^5$. Note that $\ker(\omega) = \text{Vect}\{e_5, e_6, e_7\} \subset H$ and $\ker(\omega|_H) = \ker(\omega) \oplus \mathbb{C}e_3 \subset H$.

Let $D = \mathbb{C}e \in \mathbb{P}H$. There are three cases.

- First case: $D \subset \ker(\omega)$. We have

$$\begin{aligned} [\pi^{-1}(D)] &= [\{\mathbb{C}f \in \mathbb{P}(V/D) \mid \omega(f, e) = 0\}] - [\{\mathbb{C}f \in \mathbb{P}(H/D) \mid \omega|_H(f, e) = 0\}] \\ &= [\mathbb{P}^5] - [\mathbb{P}^4] = \mathbb{L}^5. \end{aligned}$$

- Second case: $D \not\subset \ker(\omega)$ and $D \subset \ker(\omega|_H)$. In this case $\pi^{-1}(D) = \emptyset$, because

$$\{\mathbb{C}f \in \mathbb{P}(V/D) \mid \omega(f, e) = 0\} = \{\mathbb{C}f \in \mathbb{P}(H/D) \mid \omega|_H(f, e) = 0\}.$$

• Third case: $D \not\subset \ker(\omega|_H)$. We have

$$\begin{aligned} [\pi^{-1}(D)] &= [\{\mathbb{C}f \in \mathbb{P}(V/D) \mid \omega(f, e) = 0\}] - [\{\mathbb{C}f \in \mathbb{P}(H/D) \mid \omega|_H(f, e) = 0\}] \\ &= [\mathbb{P}^4] - [\mathbb{P}^3] = \mathbb{L}^4. \end{aligned}$$

Consequently

$$\begin{aligned} [U] &= [\mathbb{P} \ker(\omega)] \cdot \mathbb{L}^5 + ([\mathbb{P}H] - [\mathbb{P} \ker(\omega|_H)]) \cdot \mathbb{L}^4 \\ &= [\mathbb{P}^2] \cdot \mathbb{L}^5 + ([\mathbb{P}^5] - [\mathbb{P}^3]) \cdot \mathbb{L}^4 \\ &= ([\mathbb{P}^5] - 1) \cdot \mathbb{L}^4. \end{aligned}$$

We can repeat the argument with H . As $\omega|_F = 0$, we have

$$\begin{aligned} [\{T \in G(2, H) \mid \omega|_T = 0\}] &= [\{T \in G(2, F) \mid \omega|_T = 0\}] + [\mathbb{P} \ker(\omega|_H)] \cdot \mathbb{L}^4 \\ &= [G(2, 5)] + [\mathbb{P}^3] \cdot \mathbb{L}^4 \\ &= [\mathbb{P}^4] \cdot (\mathbb{L}^2 + 1) + [\mathbb{P}^3] \cdot \mathbb{L}^4. \end{aligned}$$

Finally, we get

$$\begin{aligned} [\pi^{-1}(\mathbb{C}\omega)] &= ([\mathbb{P}^5] - 1) \cdot \mathbb{L}^4 + [\mathbb{P}^4] \cdot (\mathbb{L}^2 + 1) + [\mathbb{P}^3] \cdot \mathbb{L}^4 \\ &= ([\mathbb{P}^5] - 1) \cdot \mathbb{L}^4 + ([\mathbb{P}^5] - \mathbb{L}^5) \cdot (\mathbb{L}^2 + 1) + (\mathbb{L}^3 + \mathbb{L}^2 + \mathbb{L} + 1) \cdot \mathbb{L}^4 \\ &= [\mathbb{P}^5] \cdot (\mathbb{L}^4 + \mathbb{L}^2 + 1) + \mathbb{L}^6. \quad \square \end{aligned}$$

A similar calculation gives the following result.

Lemma 3.5. *Let $\mathbb{C}\omega \in \mathbb{P}W \setminus Y_W$ be a closed point. Then the class of its fiber is*

$$[\pi^{-1}(\mathbb{C}\omega)] = [\mathbb{P}^5] \cdot (\mathbb{L}^4 + \mathbb{L}^2 + 1).$$

Proof of Proposition 3.2. Let $\mathbb{C}\omega_1 \in Y_W$ and $\mathbb{C}\omega_2 \in \mathbb{P}W \setminus Y_W$ be two closed points. Lemma 3.3 implies that

$$\begin{cases} [\pi^{-1}(Y_W)] = [Y_W] \cdot [\pi^{-1}(\mathbb{C}\omega_1)] \\ [\pi^{-1}(\mathbb{P}W \setminus Y_W)] = ([\mathbb{P}W] - [Y_W]) \cdot [\pi^{-1}(\mathbb{C}\omega_2)], \end{cases}$$

and consequently

$$[H] = [Y_W] \cdot [\pi^{-1}(\mathbb{C}\omega_1)] + ([\mathbb{P}W] - [Y_W]) \cdot [\pi^{-1}(\mathbb{C}\omega_2)].$$

Using Lemmas 3.4 and 3.5, we have

$$\begin{aligned} [H] &= [Y_W] \cdot ([\mathbb{P}^5] \cdot (\mathbb{L}^4 + \mathbb{L}^2 + 1) + \mathbb{L}^6) + ([\mathbb{P}^6] - [Y_W]) \cdot [\mathbb{P}^5] \cdot (\mathbb{L}^4 + \mathbb{L}^2 + 1) \\ &= [Y_W] \cdot \mathbb{L}^6 + [\mathbb{P}^6] \cdot [\mathbb{P}^5] \cdot (\mathbb{L}^4 + \mathbb{L}^2 + 1), \end{aligned}$$

which concludes the proof. $\square$

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